Warping

12 May 2015

Warping, morphing, mosaic

- *Slides from **Durand and Freeman (MIT)**, **Efros (CMU, Berkeley)**, **Szeliski (MSR)**, **Seitz (UW)**, **Lowe (UBC)***

http://szeliski.org/Book/

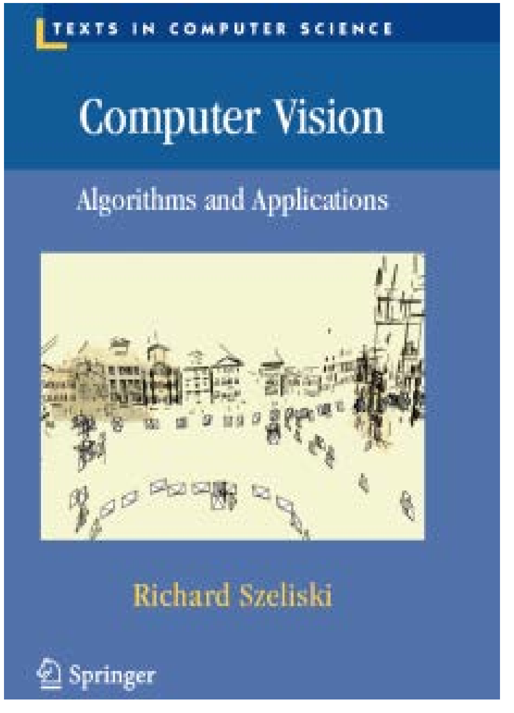
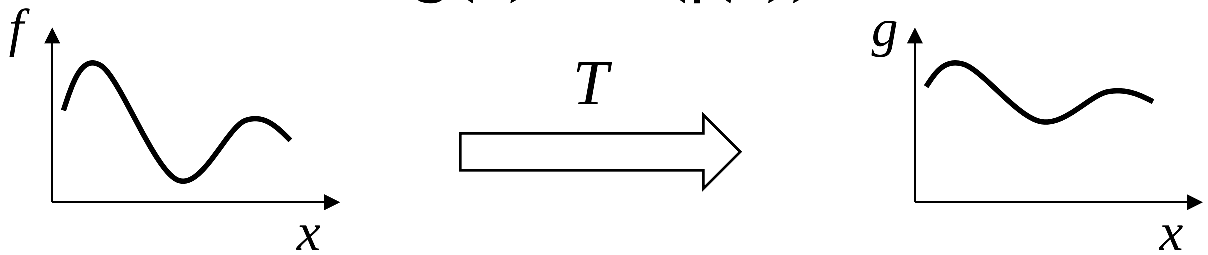


Image Warping

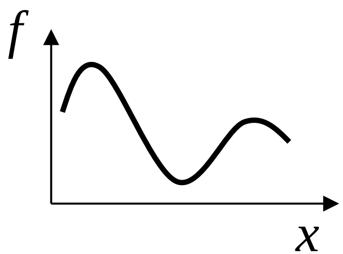
- Image filtering: change ***range*** of image

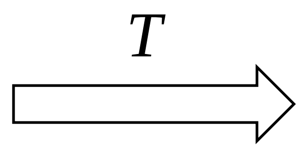
$$g(x) = T(f(x))$$



- Image warping: change ***domain*** of image

$$g(x) = f(T(x))$$





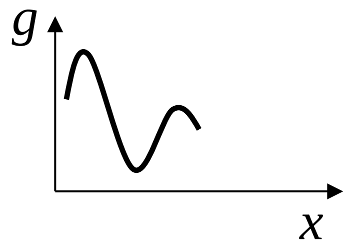
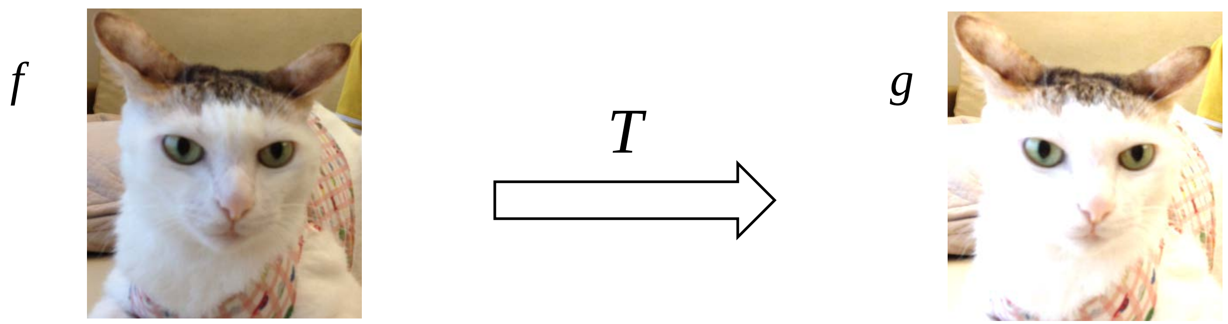
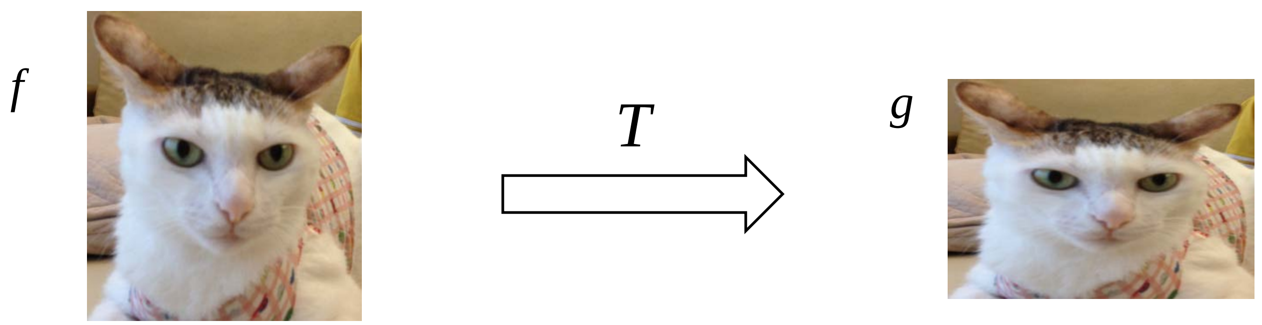


Image Warping

- Image filtering: change ***range*** of image

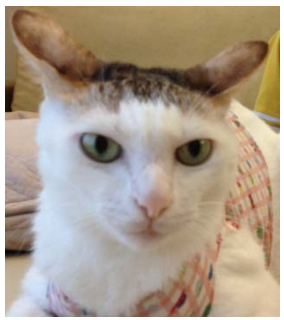


- Image warping: change ***domain*** of image



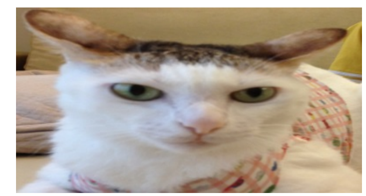
Parametric (Global) Warping

- Examples of parametric warps:



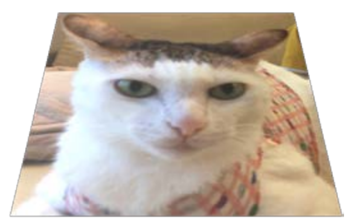
translation

rotation



aspect

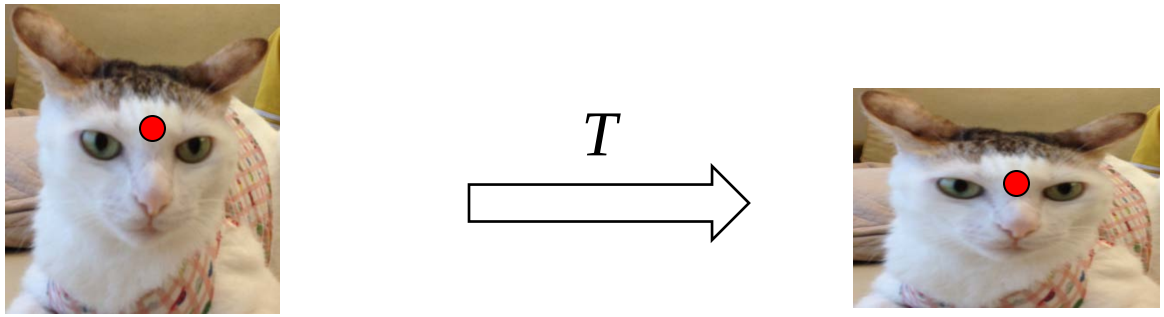
affine



projective

cylindrical

Parametric (Global) Warping



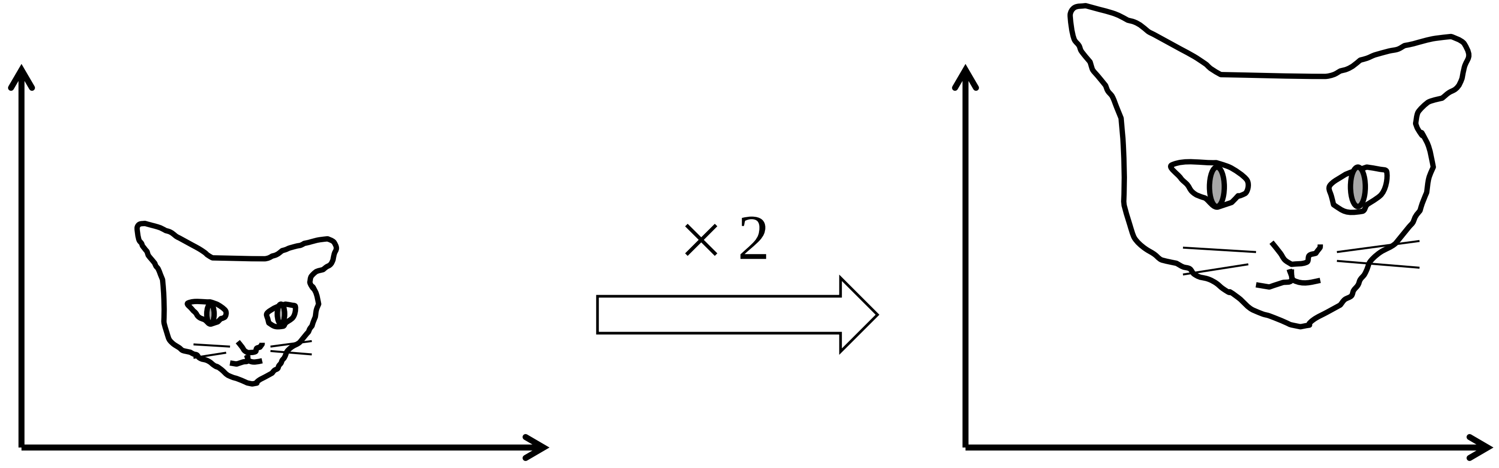
$\mathbf{p} = (x,y)$ $\mathbf{p}' = (x',y')$

- Transformation T is a coordinate-changing machine:
- $$\mathbf{p}' = T(\mathbf{p})$$
- What does it mean that T is global?
  - Is the same for any point $\mathbf{p}$
  - Can be described by just a few numbers (parameters)
- Represent T as a matrix: $\mathbf{p}' = \mathbf{M}\mathbf{p}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathrm{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

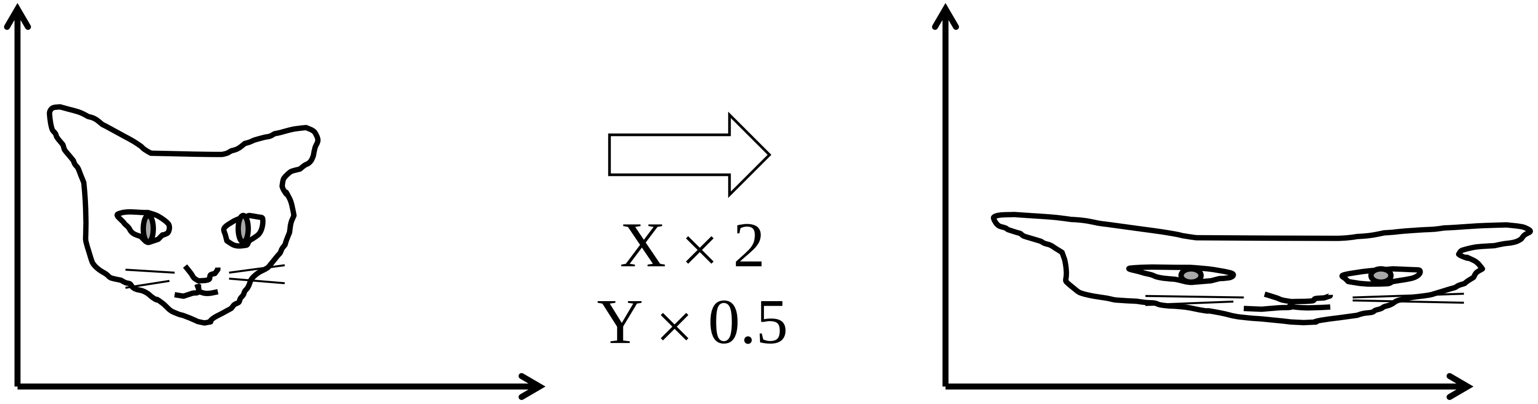
Scaling

- *Scaling* a coordinate means multiplying each of its components by a scalar
- *Uniform scaling* means this scalar is the same for all components:



Scaling

- *Non-uniform scaling*: different scalars per component:



Scaling

- Scaling operation:

$$x' = ax$$
$$y' = by$$

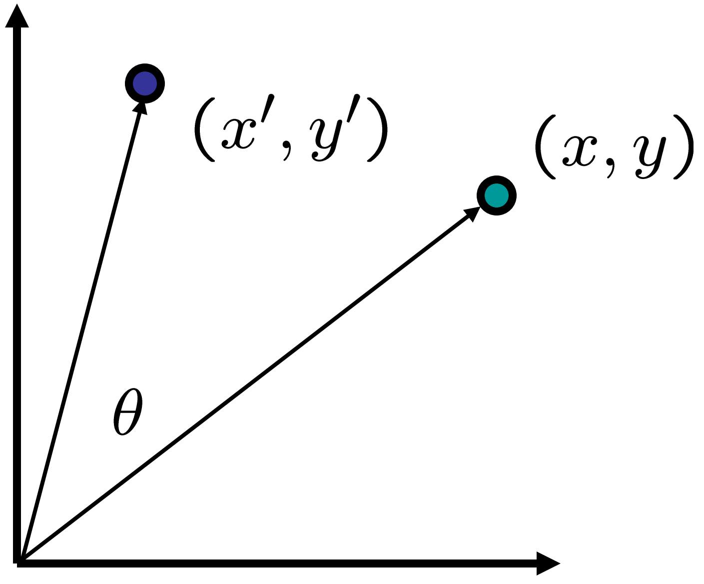
- Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

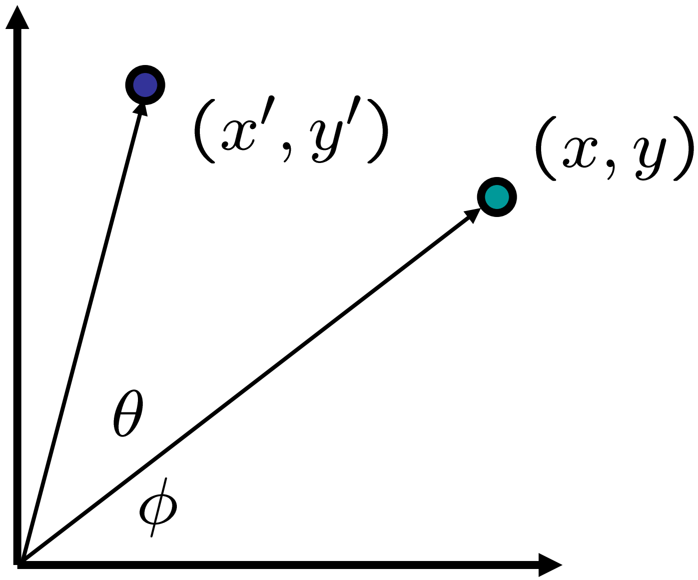
What's inverse of *S*?

2-D Rotation

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$



2-D Rotation



$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$x' = r\cos(\phi + \theta)$$
$$y' = r\sin(\phi + \theta)$$

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\sin(\phi + \theta) = \sin\phi\cos\theta + \cos\phi\sin\theta$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

2-D Rotation

- This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Even though $\cos\theta$ and $\sin\theta$ are nonlinear functions of θ,
  - ***x' is a linear combination of x and y***
  - ***y' is a linear combination of x and y***

- What is the inverse transformation?
  - Rotation by $-\theta$
  - For rotation matrices $\mathbf{R}^{-1} = \mathbf{R}^T$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$x' = x \qquad\qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = y$$

2D Scale around (0,0)?

$$x' = s_x \, x \qquad\qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = s_y \, y$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$x' = x + sh_x\, y$$
$$y' = sh_y\, x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?

$$x' = -x \qquad y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$x' = -x \qquad y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

NO!

$$\boxed{\text{Only linear 2D transformations can be represented with a 2x2 matrix}}$$

All 2D Linear Transformations

- Linear transformations are combinations of …
  - Scale
  - Rotation
  - Shear
  - Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:
  - Origin maps to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous Coordinates

- Q: How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$
$$y' = y + t_y$$

Homogeneous Coordinates

- *Homogeneous coordinates*
  - Represent coordinates in 2 dimensions with a 3-vector

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Q: How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$
$$y' = y + t_y$$

- A: Using the rightmost column:

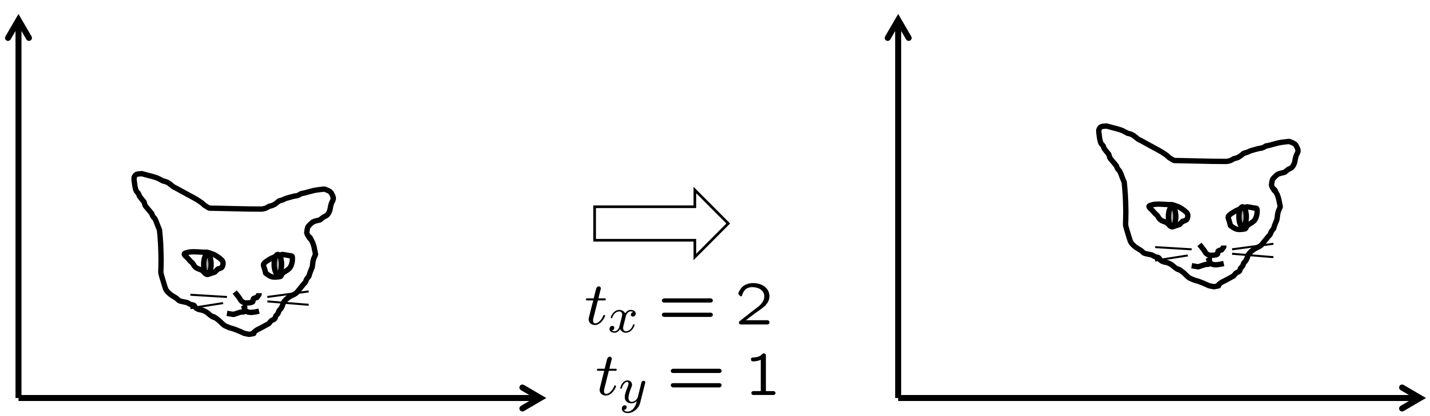
$$\text{Translation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

- Example of translation

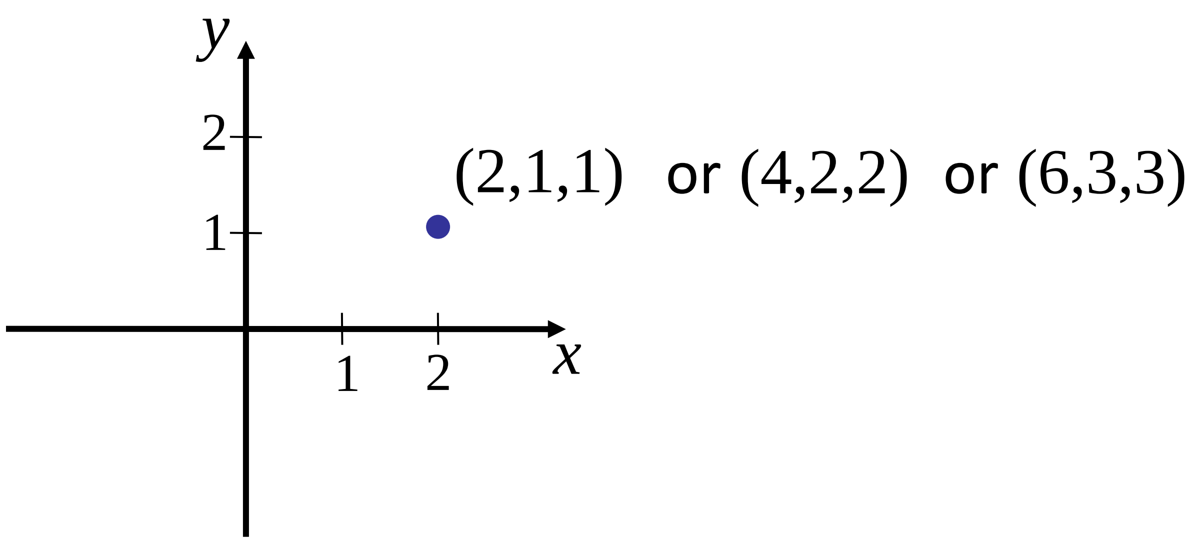
Homogeneous Coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



Homogeneous Coordinates

- Add a 3rd coordinate to every 2D point
  - (x, y, w) represents a point at location $(x/w, y/w)$
  - $(x, y, 0)$ represents a point at infinity
  - $(0, 0, 0)$ is not allowed



Convenient coordinate system to represent many useful transformations

Basic 2D Transformations

- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Affine Transformations

- Affine transformations are combinations of …
  - Linear transformations, and
  - Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition
  - Models change of basis

Projective Transformations

- Projective transformations …
  - Affine transformations, and
  - Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines do not necessarily remain parallel
  - Ratios are not preserved
  - Closed under composition
  - Models change of basis

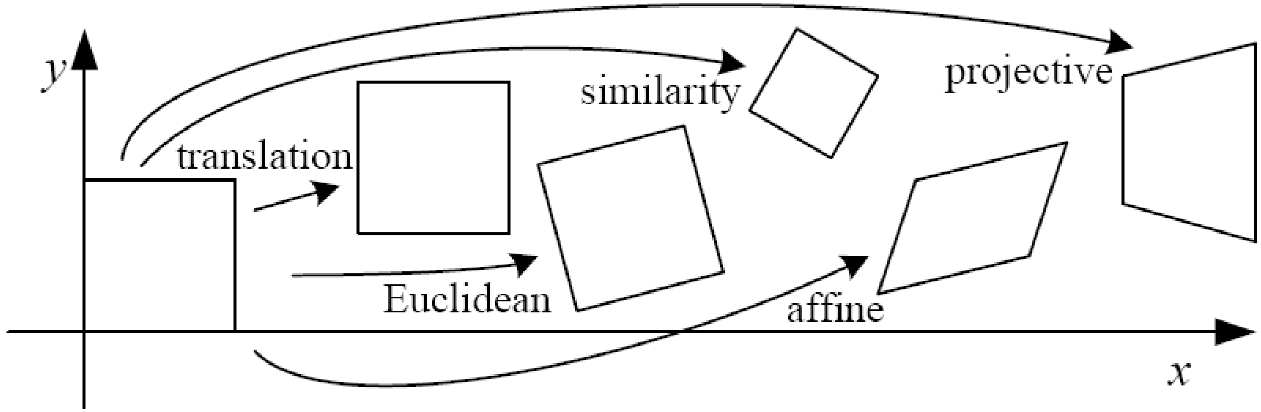
Matrix Composition

- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\mathbf{p}' \qquad T(t_x, t_y) \qquad R(\theta) \qquad S(s_x, s_y) \qquad \mathbf{p}$$

2D Image Transformations

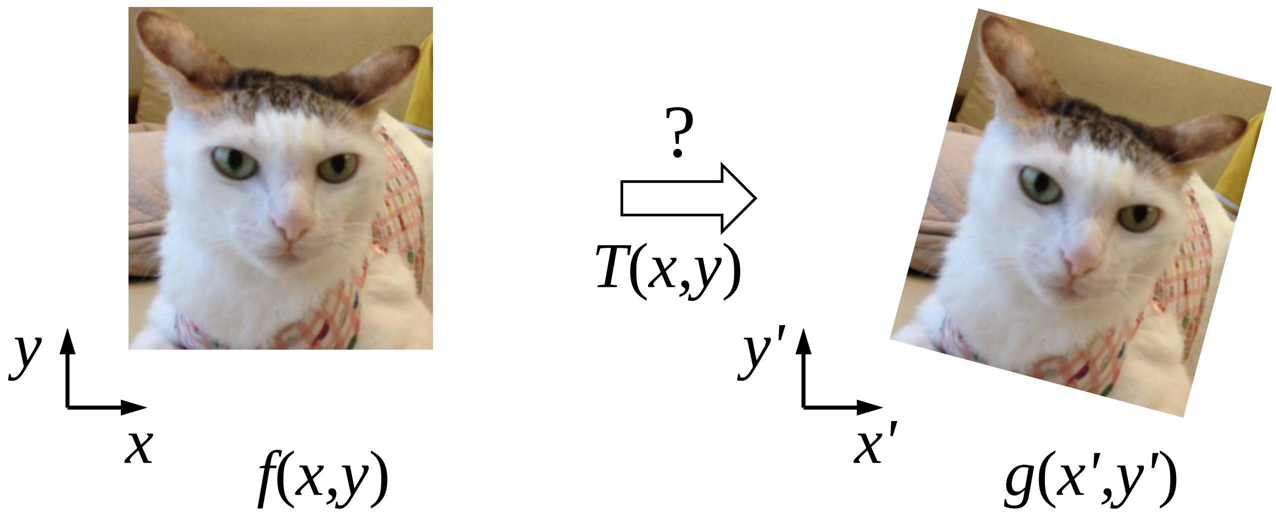


Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\left[\, \boldsymbol{I} \mid \boldsymbol{t} \,\right]_{2\times 3}$	2	orientation $+ \cdots$	
rigid (Euclidean)	$\left[\, \boldsymbol{R} \mid \boldsymbol{t} \,\right]_{2\times 3}$	3	lengths $+ \cdots$	
similarity	$\left[\, s\boldsymbol{R} \mid \boldsymbol{t} \,\right]_{2\times 3}$	4	angles $+ \cdots$	
affine	$\left[\, \boldsymbol{A} \,\right]_{2\times 3}$	6	parallelism $+ \cdots$	
projective	$\left[\, \tilde{\boldsymbol{H}} \,\right]_{3\times 3}$	8	straight lines	

Closed under composition and inverse is a member

Morphing

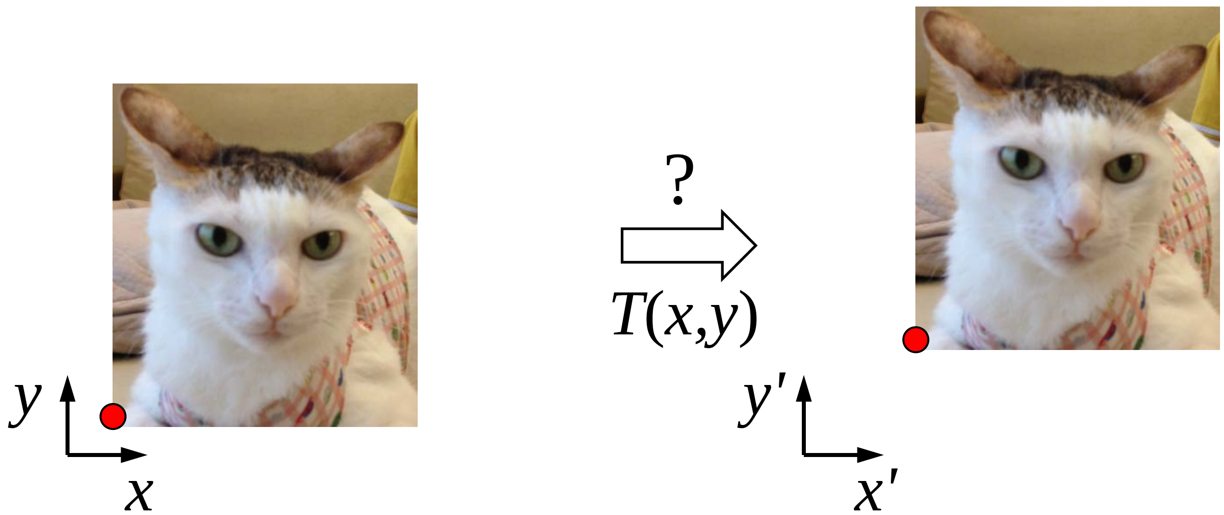
Recovering Transformations



- What if we know f and g and want to recover the transform T?
  - Let user provide correspondences
    - How many do we need?

[Efros]

Translation: # Correspondences?

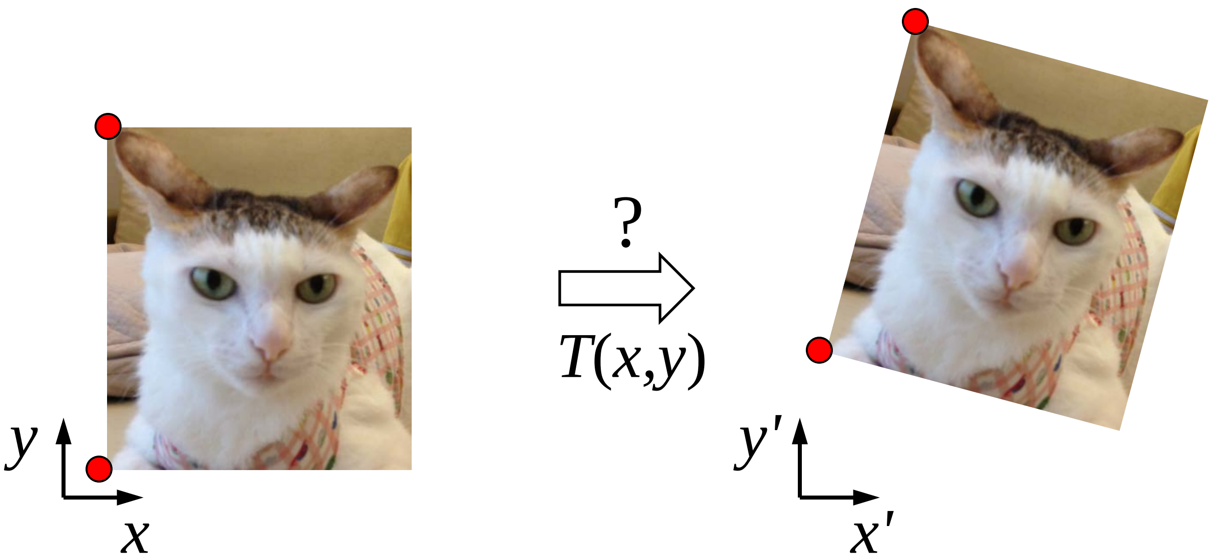


- How many correspondences needed for translation?
- How many Degrees of Freedom?
- What is the transformation matrix?

$$M = \begin{bmatrix} 1 & 0 & p'_x - p_x \\ 0 & 1 & p'_y - p_y \\ 0 & 0 & 1 \end{bmatrix}$$

[Efros]

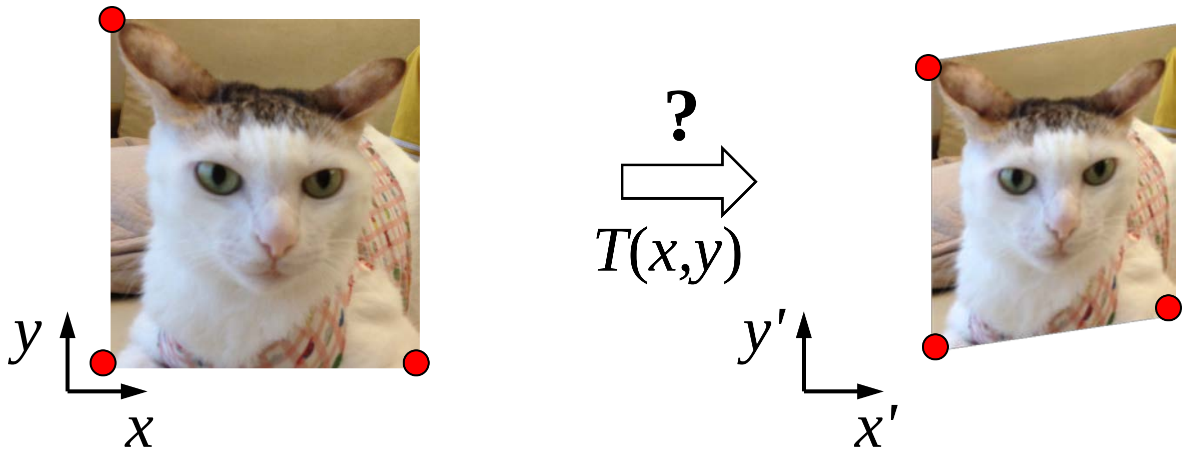
Euclidian: # Correspondences?



- How many correspondences needed for translation + rotation?
- How many DOF?

[Efros]

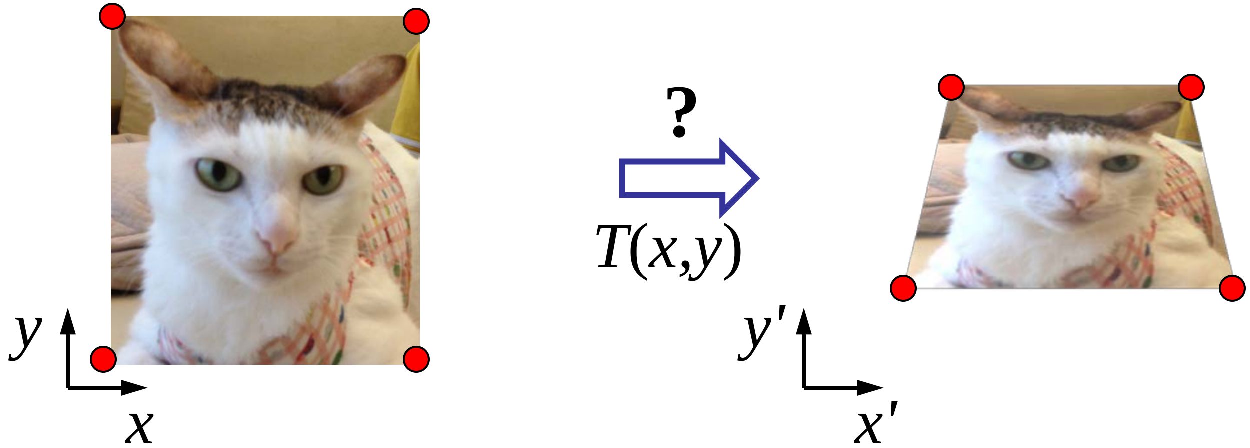
Affine: # Correspondences?



- How many correspondences needed for affine?
- How many DOF?

[Efros]

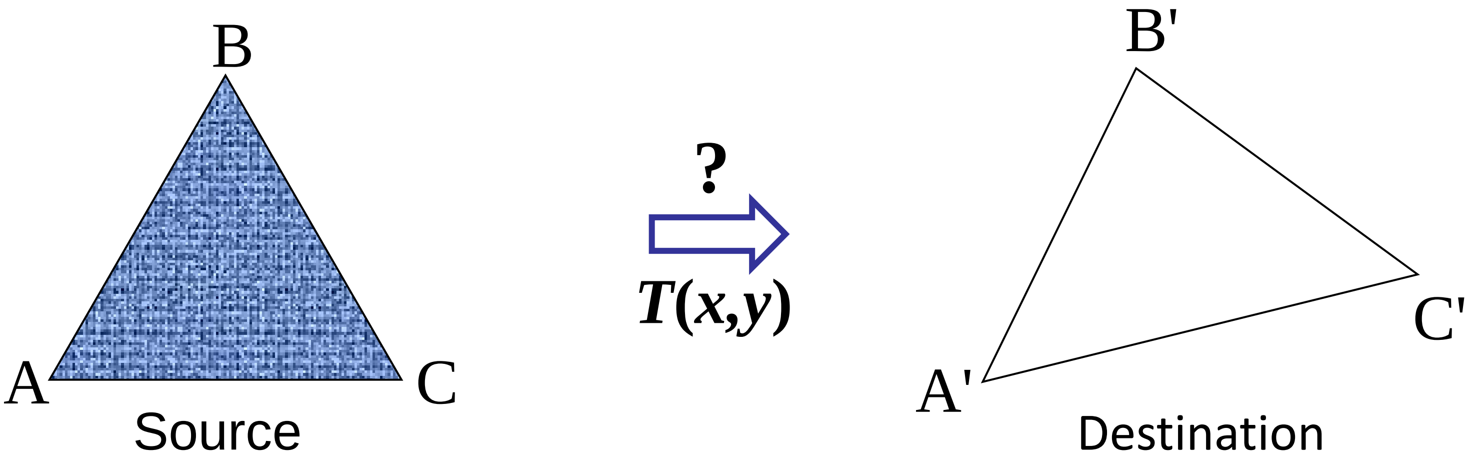
Projective: # Correspondences?



- How many correspondences needed for projective?
- How many DOF?

[Efros]

Example: Warping Triangles

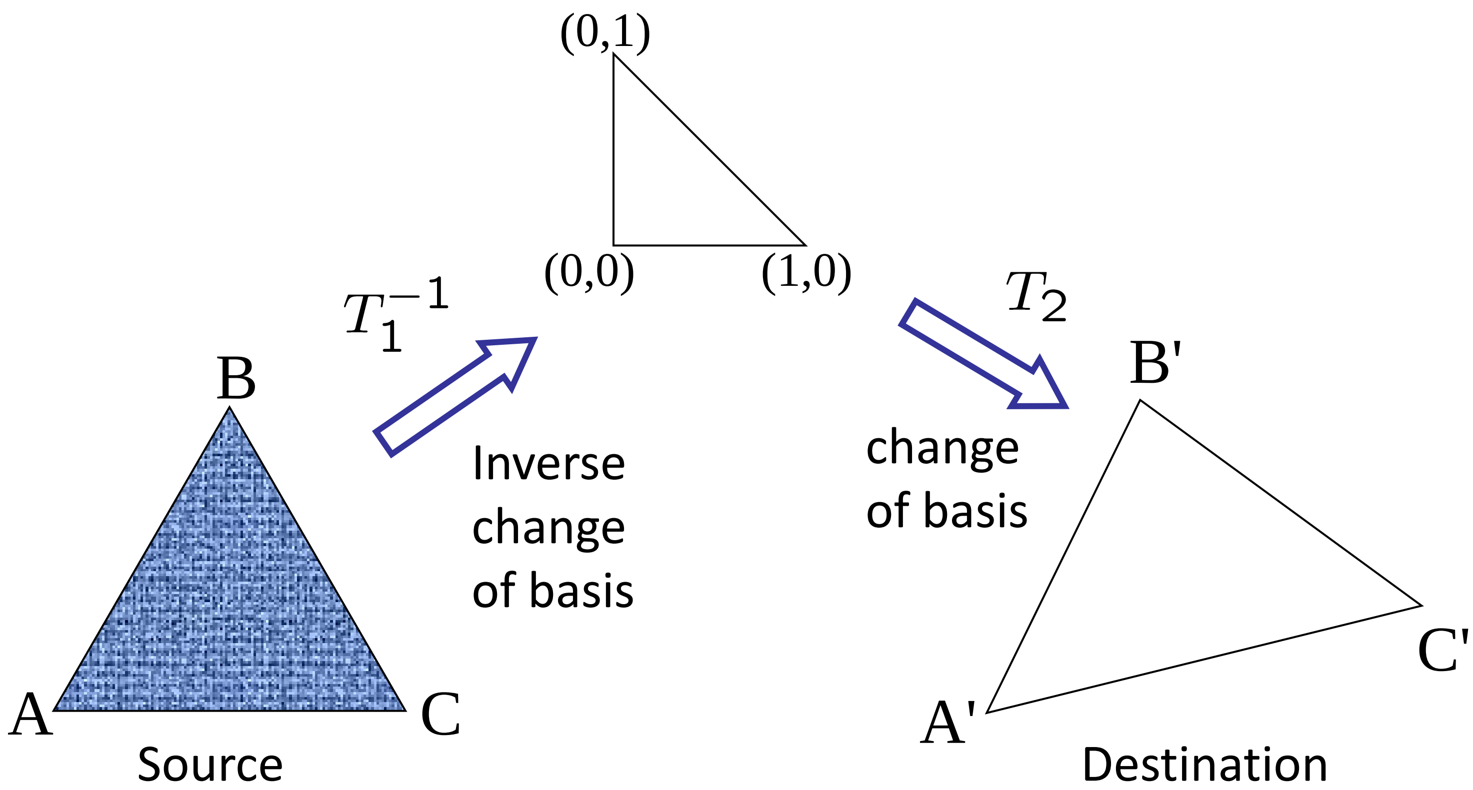


- Given two triangles: ABC and A'B'C' in 2D (12 numbers)
- Need to find transform T to transfer all pixels from one to the other.
- What kind of transformation is T?
- How can we compute the transformation matrix:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

[Efros]

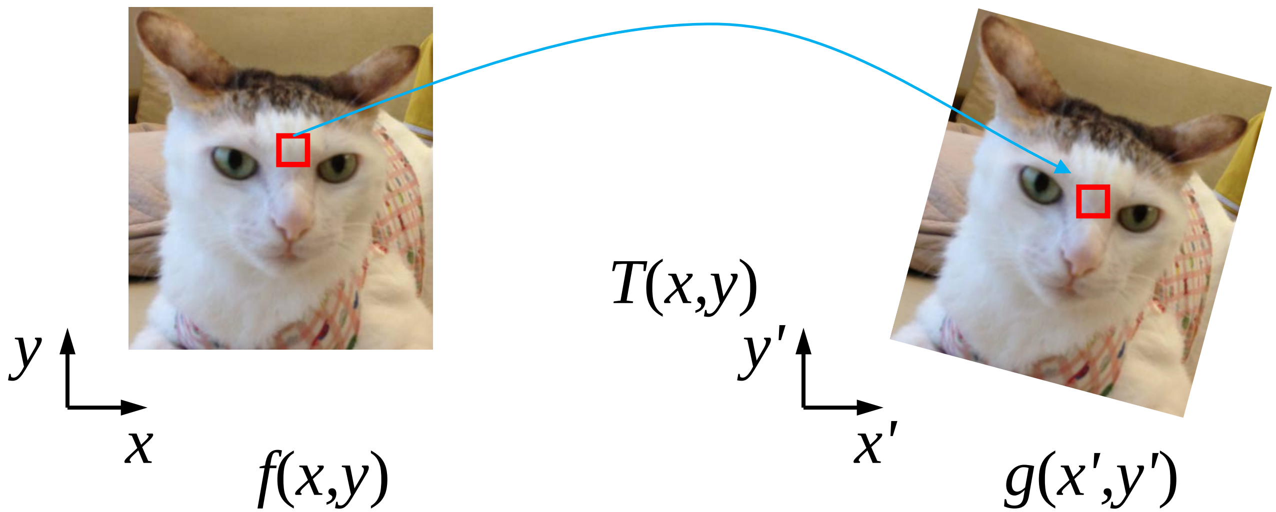
Warping Triangles



Don’t forget to move the origin too!

[Efros]

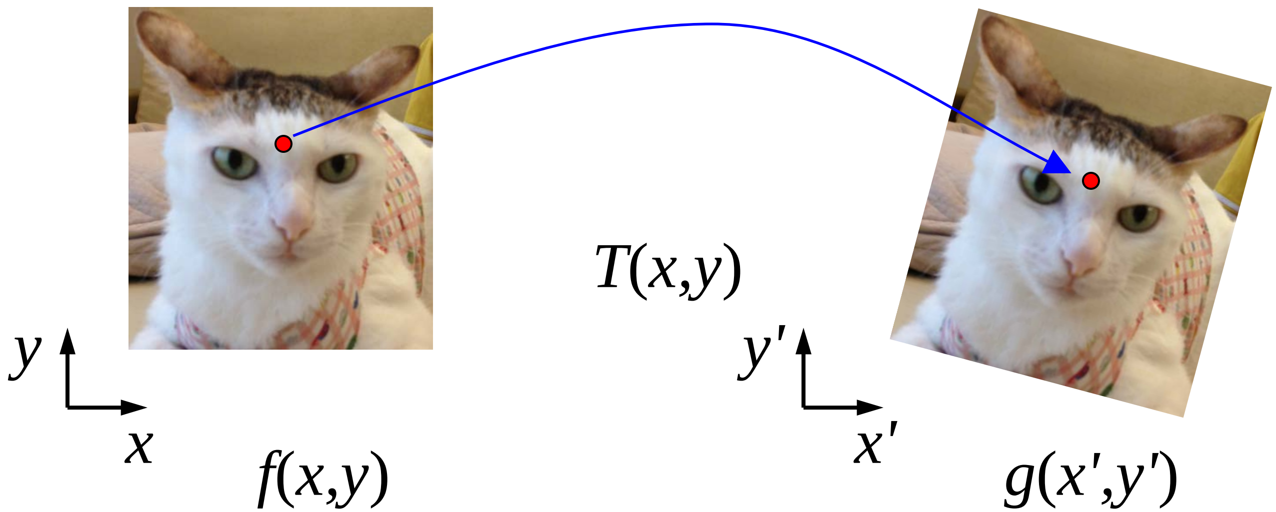
Image Warping



- Given a coordinate transform $(x',y') = T(x,y)$ and a source image $f(x,y)$, how do we compute a transformed image $g(x',y') = f(T(x,y))$?

[Efros]

Forward Warping

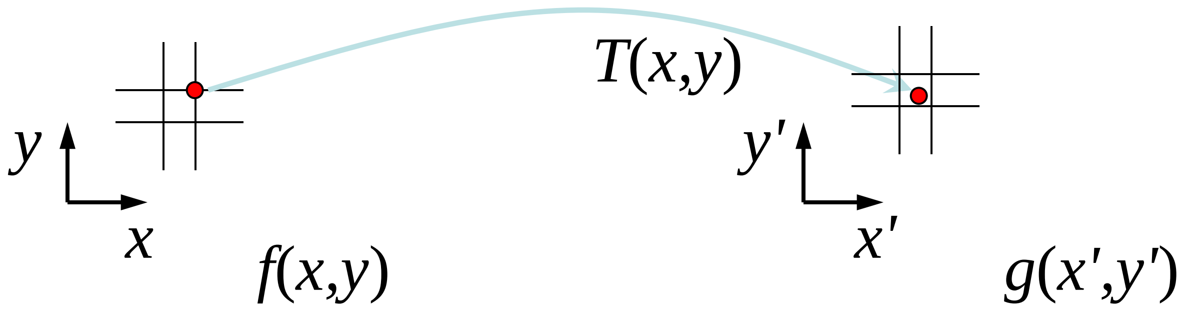


- Send each pixel $f(x,y)$ to its corresponding location
  $(x',y') = T(x,y)$ in the second image

Q: What if pixel lands "between" two pixels?

[Efros]

Forward Warping



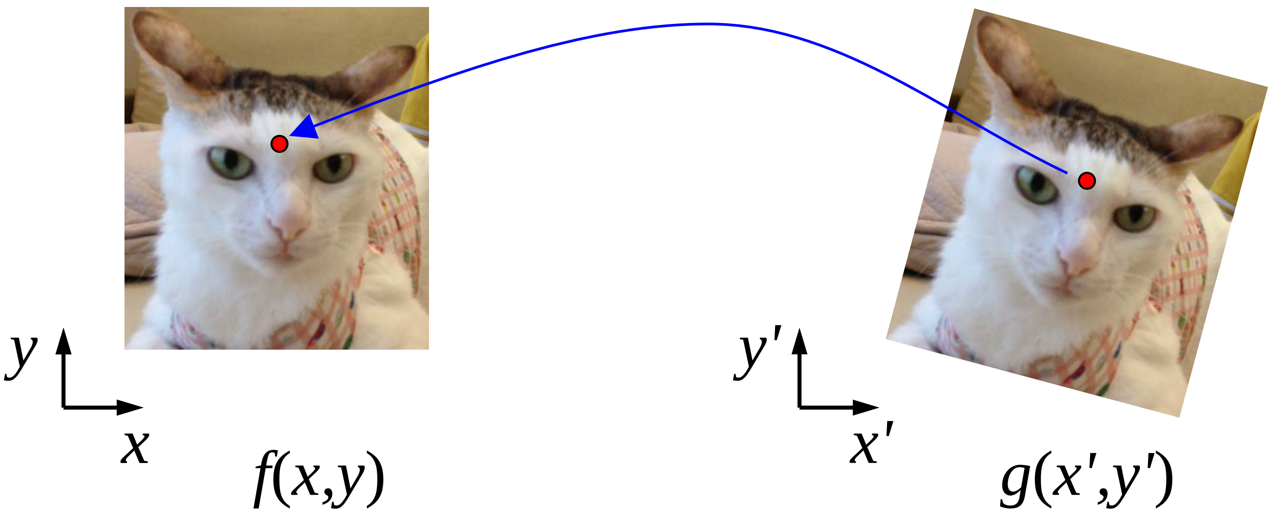
- Send each pixel $f(x,y)$ to its corresponding location
- $(x',y') = T(x,y)$ in the second image

Q: What if pixel lands "between" two pixels?

A: Distribute color among neighboring pixels (x',y')

[Efros]

Inverse Warping

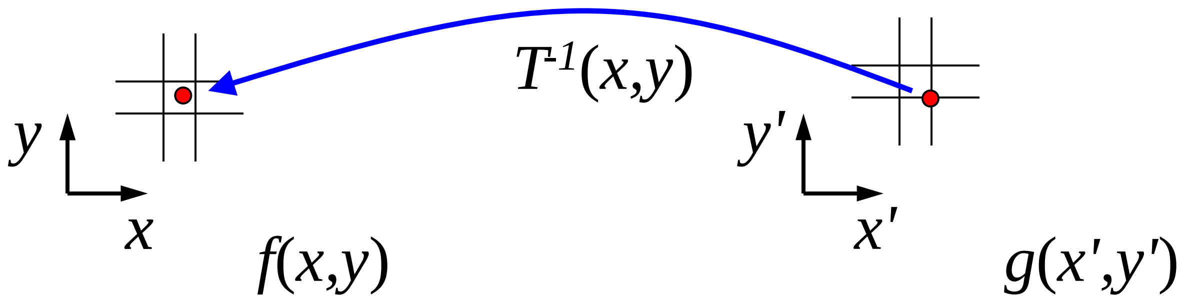


- Get each pixel $g(x',y')$ from its corresponding location $(x,y) = T^{-1}(x',y')$ in the first image

Q: What if pixel comes from "between" two pixels?

[Efros]

Inverse Warping



- Get each pixel $g(x',y')$ from its corresponding location $(x,y) = T^{-1}(x',y')$ in the first image

Q: What if pixel comes from "between" two pixels?

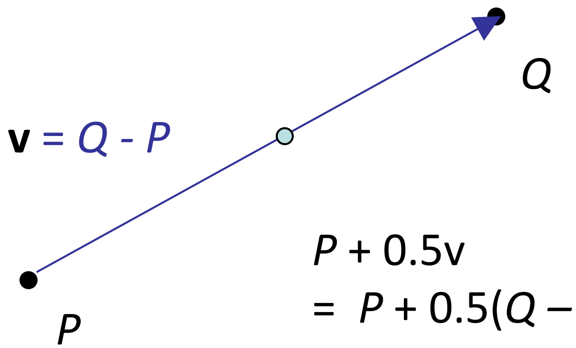
A: *Interpolate* color value from neighbors

- nearest neighbor, bilinear, Gaussian, bicubic

[Efros]

Linear Interpolation

What's the average of P and Q?



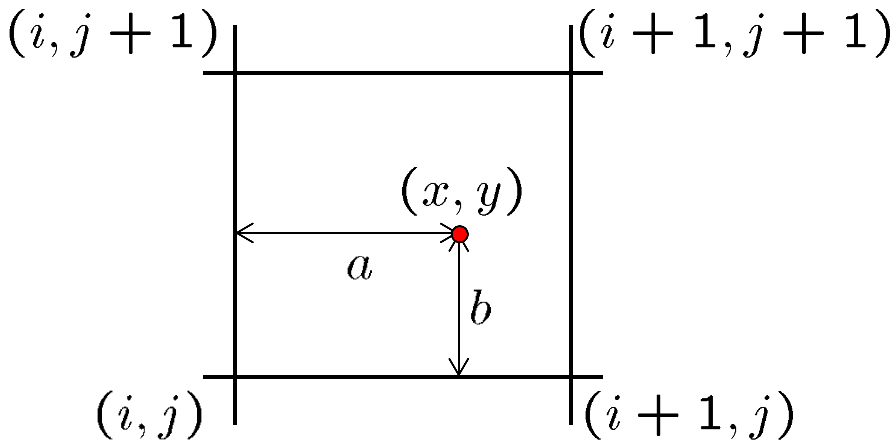
$P + 0.5v$
$= P + 0.5(Q - P)$
$= 0.5P + 0.5\,Q$

Linear Interpolation
(Affine Combination):
New point $aP + bQ$,
defined only when $a+b = 1$
So $aP+bQ = aP+(1-a)Q$

[Efros]

Bilinear Interpolation

- Sampling at $f(x,y)$:



$$
\begin{aligned}
f(x,y) = \;& (1-a)(1-b) && f[i,j] \\
& +a(1-b) && f[i+1,j] \\
& +ab && f[i+1,j+1] \\
& +(1-a)b && f[i,j+1]
\end{aligned}
$$

[Efros]

Forward vs. Inverse Warping

- Q: Which is better?
- A: Usually inverse—eliminates holes
  - However, it requires an invertible warp function—not always possible...

[Efros]